

Absence of self-organized criticality in a random-neighbor version of the OFC stick-slip model

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We report some numerical simulations to investigate the existence of a self-organized critical (SOC) state in a random-neighbor version of the OFC model for a range of parameters corresponding to a non-conservative case. In contrast to a recent work, we do not find any evidence of SOC. We use a more realistic distribution of energy among sites to perform some analytical calculations that agree with our numerical conclusions.

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The first cellular automaton to display SOC (self-organized criticality), the sandpile model [1], required both a bulk conservation law and a lattice structure with open boundary conditions, that were thought to be the main ingredients to allow the building of the spatial correlations [2,3]. There are some attempts to construct much simpler and more tractable mean-field versions of these systems [4–6]. Olami, Feder and Christensen [7] introduced a non-conservative cellular automaton (the earthquake model) that displays self-organized critical behavior for a certain range of a parameter α related to the breaking of a conservation law. A detailed analysis of this model [8,9] seems to show that the spatial inhomogeneities created by the lattice, the particular boundary conditions, and the global driven mechanism, are the essential ingredients for the existence of SOC. In a recent publication, Lise and Jensen

[10] considered a random-neighbor mean-field version of the OFC model with coordination $q = 4$. According to these calculations, it is remarkable that SOC can still be observed in non-conservative cases, for $2/9 = \alpha^* < \alpha < \alpha_c = 1/4$.

In this work we revisit the random-neighbor version of the OFC model, with an arbitrary coordination q (that should be of interest in the context of the investigations of SOC in artificial neural networks [11], in which case the existence of a regular lattice is not a realistic assumption). Unlike the conclusions of Lise and Jensen, we do not find any evidence of SOC, except in the conservative case. We show (see figure 1) that the use of lattices that are not big enough to allow the occurrence of the largest avalanches can lead to the wrong impression that the system does display SOC. Of course, this is an important word of caution about all numerical evidences of SOC. Within the mean-field approach, we show that the use of a simple and slightly more realistic assumption about the form of the stationary probability distribution $p(E)$ of the energies of the sites is enough to completely change the conclusions of Lise and Jensen. Using this simplified form of $p(E)$, there is just a very small range of parameters where SOC might occur in the non-conservative context.

The random-neighbor version of the OFC model consists of N sites with an energy $E_i < E_c$, for $i = 1, \dots, N$, where E_c is a threshold value (the energies of the stable sites will be labelled by a superscript $-$). During a long time scale, there is a continuous build up of all energies until the energy of a certain site i reaches the value E_c . Then, site i becomes unstable (its energy being labelled by a superscript $+$), and the system relaxes in a very short time scale. The unstable site i transfers an energy αE_+ ($\alpha \leq 1/q$) to q random neighbors, which may also become unstable and may generate an avalanche that only stops when the energies of all sites are again below E_c . The equality $\alpha = \alpha_c = 1/q$ corresponds to the conservative limit.

To make contact with the work of Lise and Jensen [10], we define a "re-scaled" dissipation constant, $\alpha_q = q\alpha/4$. According to the same approximate calculations of Lise and Jensen, the critical value α_q^* below which the avalanches have a typical size is given by $\alpha_q^* = q/(4q + 2)$. If $q = 4$, we have $\alpha_4^* = 2/9 = 0.222$, as obtained by Lise and Jensen.

Our own simulations, however, indicate that the density distribution of the avalanche sizes $\rho(s)$ converges to a fixed curve that does not depend on the size of the lattice, even for values of α_q far beyond α_q^* , given by our mean-field calculations (see figures 1 and 2 for $\rho(s)$ and the average avalanche size, $\langle s \rangle$, respectively). Because of practical limitations, it is not feasible to simulate systems closer to the conservative limit, but we strongly believe that the collapse of $\rho(s)$ to a universal curve, and the convergence of $\langle s \rangle$ to typical value, regardless of the size of the lattice, for all values of α_q that we were able to use in the simulations, do indicate that the system is not in a critical state.

The results of numerical simulations for $p(E)$, with $q = 4$, are shown in figure 3 (for simplicity we assume from now on that $\alpha_4 = \alpha$). These results clearly show that $p(E)$ is not a simple constant. We then decided to use the same approach of Lise and Jensen [10], but supposing that the distribution $p(E)$ has the (more realistic) form shown in the detail of figure 3, where Δ_p is half the width of each peak, Δ_b is the width of the gaps between two peaks, and $E^* = 3\Delta_b + 7\Delta_p$ is the maximum value of E for which $p(E) \neq 0$. We then have

$$P_+(E^+) = \frac{\int_{E_c - \alpha E^+}^{E_c} p(E) dE}{\int_0^{\infty} p(E) dE} = \frac{1}{7a\Delta_p} \int_{E_c - \alpha E^+}^{E_c} p(E) dE. \quad (1)$$

The lower limit of the integral in the numerator, $E_c - \alpha E^+$, can belong to any of the intervals that define the four peaks of the distribution, to which we assign the indices $i = 1, 2, 3, 4$. Now we have to consider each one of these possibilities (to simplify the notation, we use the superscript i to refer to the value of $E_c - \alpha E_+$). The integrals $P_+^i(E^+)$ have the generic form

$$P_+^i(E^+) = 1 + \frac{(i-1)\Delta_b}{7\Delta_p} - \frac{E_c}{7\Delta_p} + \frac{\alpha E^+}{7\Delta_p}. \quad (2)$$

Therefore, the branching rate σ is given by

$$\sigma = 4 P_+^i = 4 \left[1 + \frac{(i-1)\Delta_b}{7\Delta_p} - \frac{E_c}{7\Delta_p} + \frac{\alpha \langle E^+ \rangle}{7\Delta_p} \right]. \quad (3)$$

Now, we write the mean-value of the energy of an unstable site,

$$\langle E^+ \rangle^i = \frac{\langle E^- \rangle^i}{(1 - \alpha)}, \quad (4)$$

where

$$\langle E^- \rangle = \frac{\int_{E_c - \alpha \langle E^+ \rangle}^{E_c} E p(E) dE}{\int_{E_c - \alpha \langle E^+ \rangle}^{E_c} p(E) dE}. \quad (5)$$

Therefore,

$$\langle E^+ \rangle^i = \frac{E_c}{\alpha(2 - \alpha)} - \frac{[7\Delta_p + (i - 1)\Delta_b](1 - \alpha)}{\alpha(2 - \alpha)} \pm \frac{\sqrt{y_i}}{2\alpha(2 - \alpha)}, \quad (6)$$

where

$$y_i = 4 \{ E_c (1 - \alpha) - [7\Delta_p + (i - 1) \Delta_b] \}^2 + 4\alpha (2 - \alpha) [x_i - 14(i - 1)] \Delta_p \Delta_b, \quad (7)$$

with $x_i = 24, 26, 32$, and 42 , for $i = 1, 2, 3$, and 4 , respectively.

To reach a critical state, we have to impose $\sigma \geq 1$. Taking $\sigma = 1$, and using Eq. (6), instead of Lise and Jensen's approximate result, we have

$$7\Delta_p (2 + \alpha) + 4(i - 1) \Delta_b - 4E_c (1 - \alpha) + 7\alpha E_c \pm 2\sqrt{y_i} = 0. \quad (8)$$

For instance, if we take $\Delta_p = 0.08$ and $\Delta_b = 0.1$, the critical branching condition leads to values of α^* outside of the physical range (that is, $\alpha^* > 1/4$). Therefore, in this particular case, it is physically forbidden to assume that $\sigma = 1$, so there is no self-organized critical state.

In the conservative limit (for $\Delta_p \rightarrow 0$, $\Delta_b \rightarrow \alpha E_c$), the four peaks of $p(E)$ tend to four delta functions at $(i - 1) \alpha E_c$. In this case, it is easy to see that $\sigma = 1$ leads to the only possibility $\alpha^* = \alpha_c = 1/4$ (we obtain $\alpha^* > 1/4$ for $i = 1, 2$, and 3). It can also be shown that, if we consider the limit $\Delta_b \rightarrow 0$ and $\Delta_p \rightarrow E_c / 7$, which corresponds to the approximation of Lise and Jensen, then $\alpha^* = 2/9$.

In general, from Eq. (8), for all values of i , the regions of parameters associated with $\alpha \leq 1/4$ are determined by

$$E_c - \frac{175}{24}\Delta_p - \frac{2x_i}{21}\Delta_b \leq 0. \quad (9)$$

From this inequality, and the relation $E_c \geq 7\Delta_p + 3\Delta_b$ (see figure 3), we see that only in a very small region of the parameters Δ_p and Δ_b (see figure 4) there are values of α^* in the physical range (that is, such that $0 < \alpha^* \leq 1/4$). In all of those cases, Δ_b is very small, and the shape of $p(E)$ is very close to the constant form used by Lise and Jensen.

In conclusion, on the basis of a mean-field argument, supplemented by a more realistic approximation for the distribution of energies $p(E)$, we give some analytical indications to support our own numerical findings that the random-neighbor version of the OFC stick-slip model cannot display SOC outside of the conservative limit. At the time we were writing this paper, we came to know about two other works [12,13] that lead essentially to the same conclusions.

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Figure Captions

Fig. 1- a) Distribution of the avalanche sizes (number of topples) for $q = 4$, $\alpha = 0.23$, and different lattice sizes, L . The number of iterations is $n = 2 \times 10^6$. For $L \geq 400$, the curves collapse to the same form, which indicates that there is no self-organized critical state; b) Using $L = 100$, $\alpha = 0.23$, and $n = 4 \times 10^6$, the statistics is similar for different coordinations, $q = 4, 6$, and 10 . However, for large q , and α not too large, the redistribution of energy is not sufficient to generate larger avalanches.

Fig. 2- a) Average avalanche size $\langle s \rangle$ versus lattice size L , for $q = 4$, and different values of α_q (smaller and larger than Lise and Jensen's value, $\alpha_q^* = 2/9$). For $\alpha_q^* = 0.23$, for example, using bigger lattice sizes ($L = 600$), we can see the exponential behavior of $\langle s \rangle$ (in contrast to Lise and Jensen's simulations). For $\alpha_q \geq 0.24$, it is necessary to use much bigger lattice sizes to see this type of behavior; b) Average avalanche sizes versus lattice size for $\alpha_q = 0.23$, and different coordination numbers, $q = 2, 4$, and 6 . For larger values of q , it is sufficient to use a small lattice size to see the exponential behavior of $\langle s \rangle$.

Fig 3-a) Distribution of energy per site $p(E)$ versus energy E , for $q = 4$, and $\alpha_q = 0.21$, 0.22 , and 0.23 . The width of the peaks decreases as α_q increases; b) Special form of $p(E)$ as used in the calculations, where Δ_p is the half-width of the peaks, Δ_b is the width of the gaps, a is the value at the peak, and $E_c \geq E^* = 7\Delta_p + 3\Delta_b$.

Fig. 4-Space of parameters of $p(E)$ in terms of $\gamma_p = \Delta_p/E_c$ and $\gamma_b = \Delta_b/E_c$. The shaded regions correspond to the intersection between $\alpha \leq 1/4$ and $(7\Delta_p + 3\Delta_b)/E_c = 7\gamma_p + 3\gamma_b \leq 1$. Depending on the value of $E_c - \alpha E^+$, we have (a) $E_c - \alpha E^+ \in [0, \Delta_p]$; (b) $E_c - \alpha E^+ \in [\Delta_p + \Delta_b, 3\Delta_p + \Delta_b]$; (c) $E_c - \alpha E^+ \in [3\Delta_p + 2\Delta_b, 5\Delta_p + 2\Delta_b]$; and (d) $E_c - \alpha E^+ \in [5\Delta_p + 3\Delta_b, 7\Delta_p + 3\Delta_b]$.

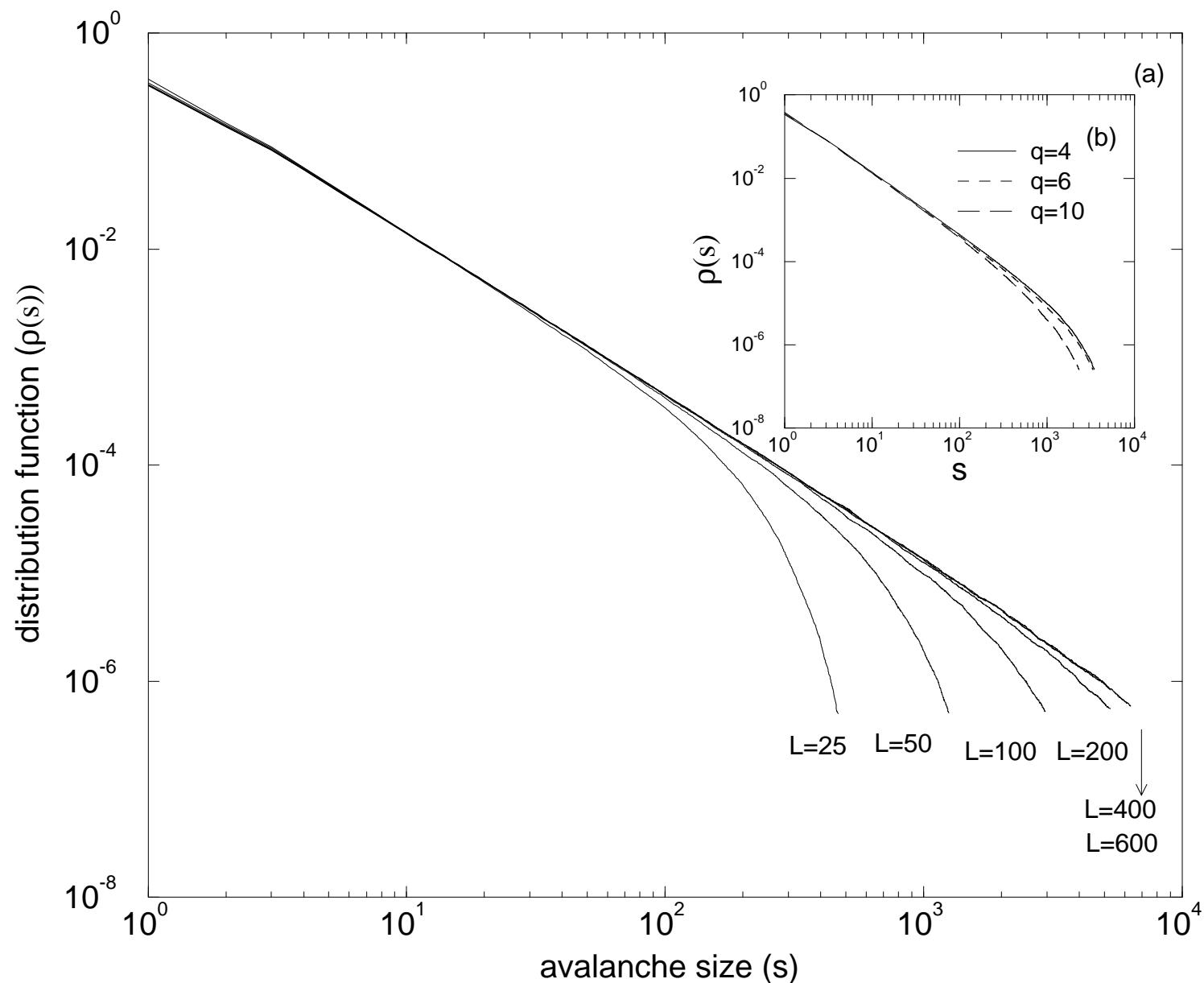


Figure 1
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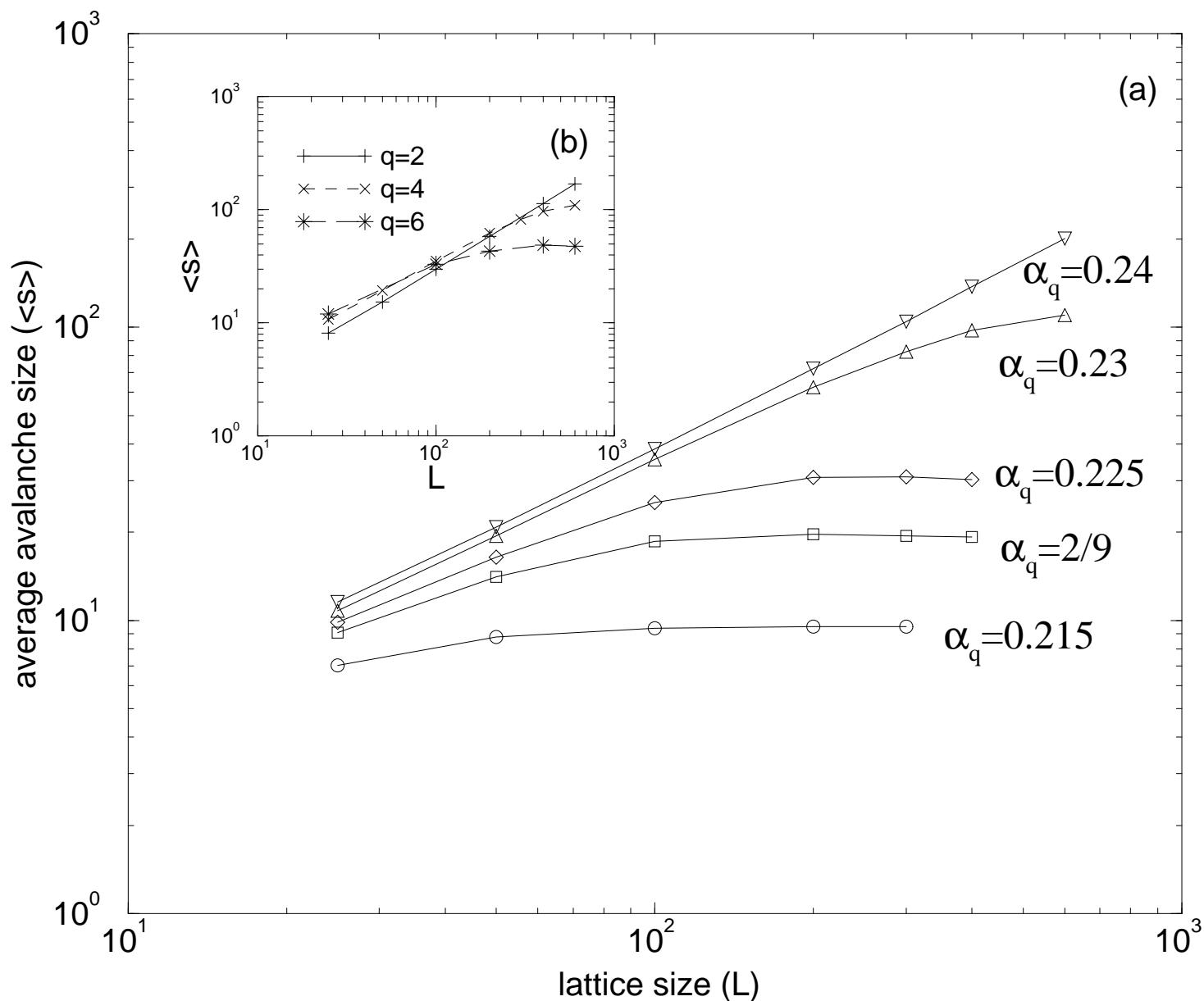


Figure 2
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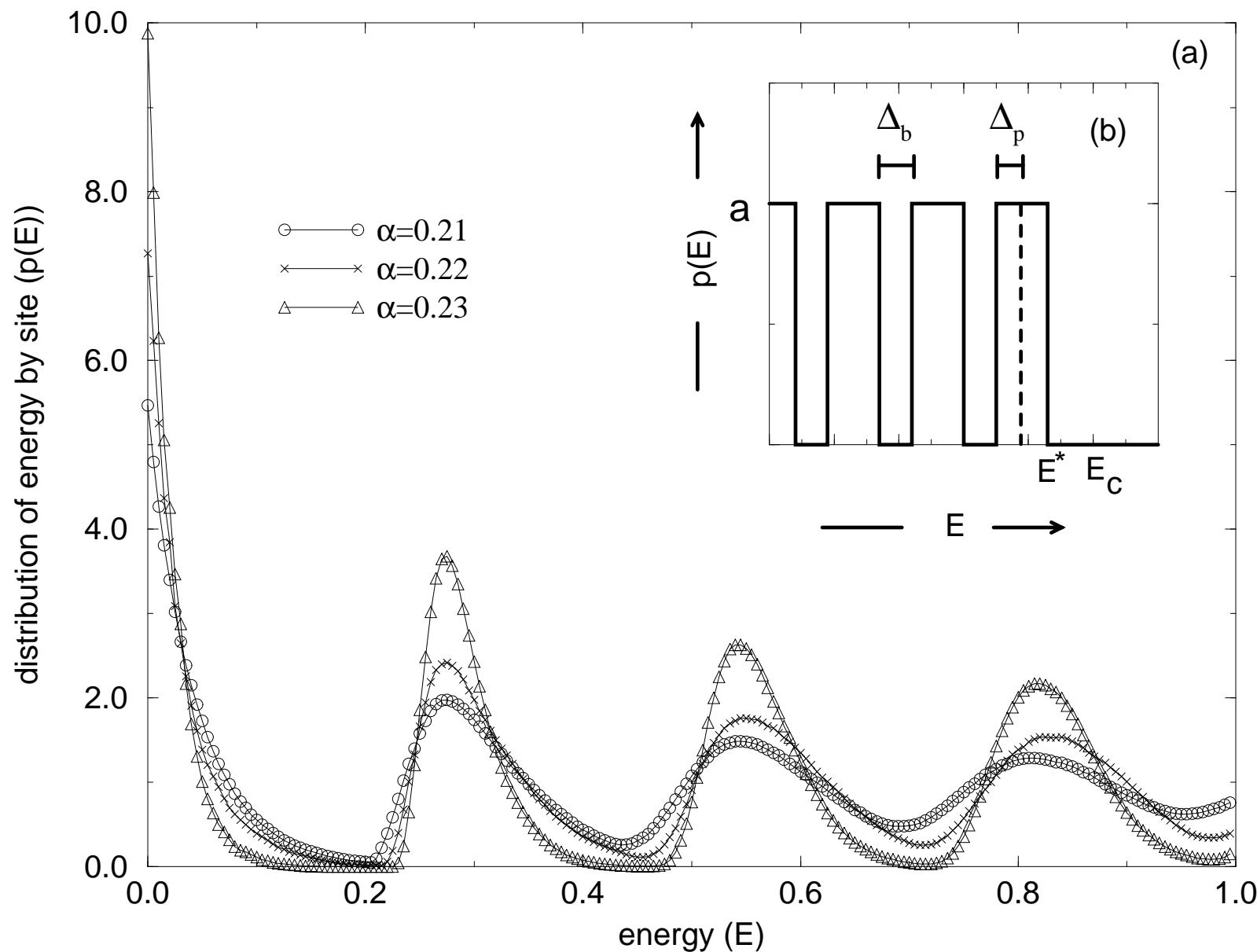


Figure 3
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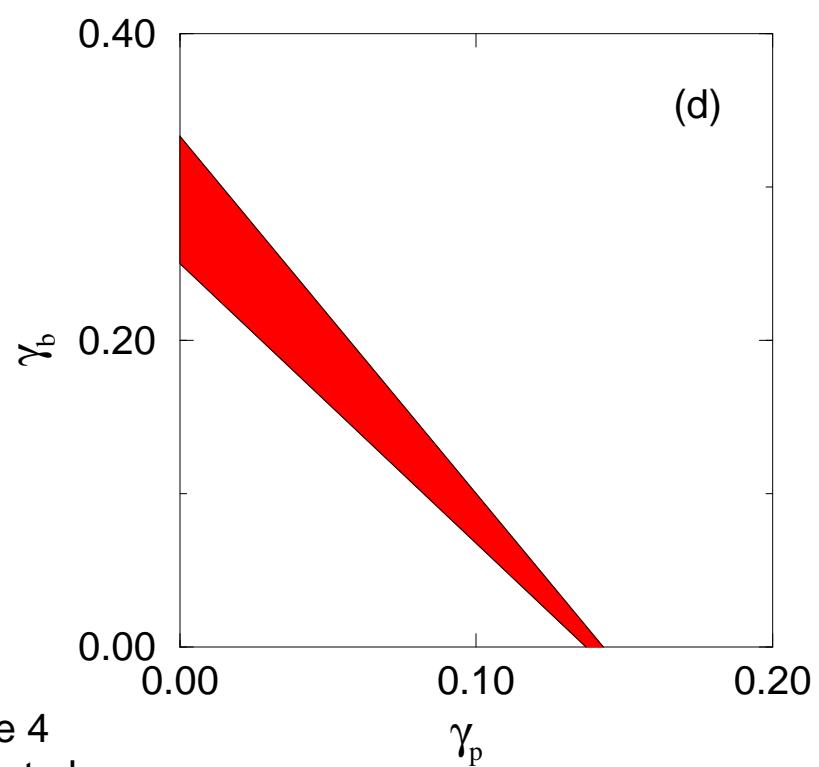
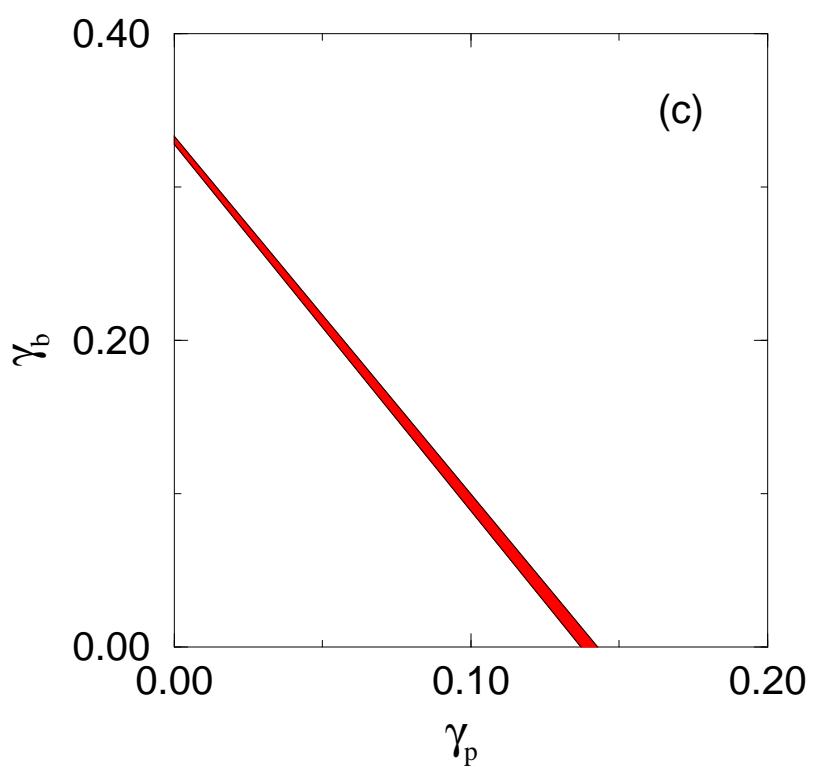
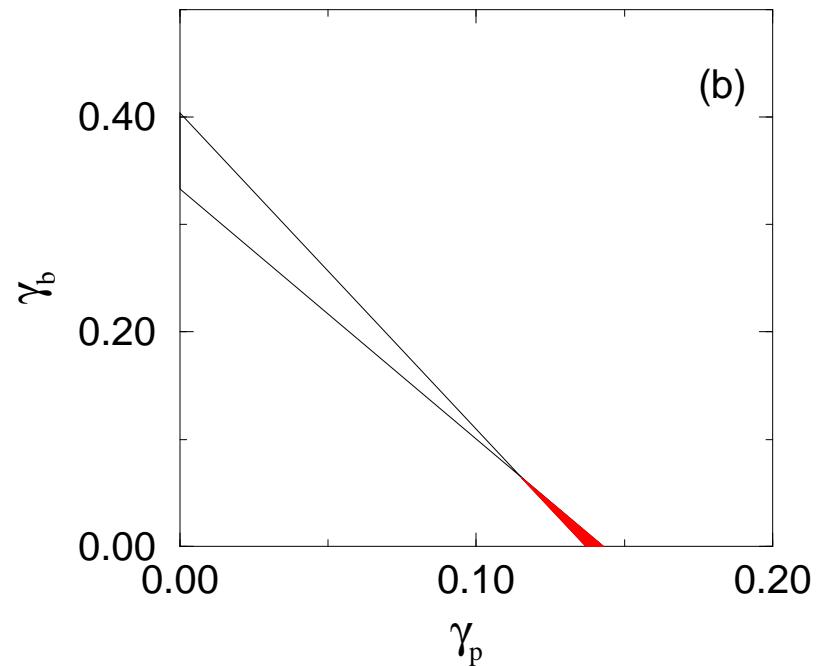
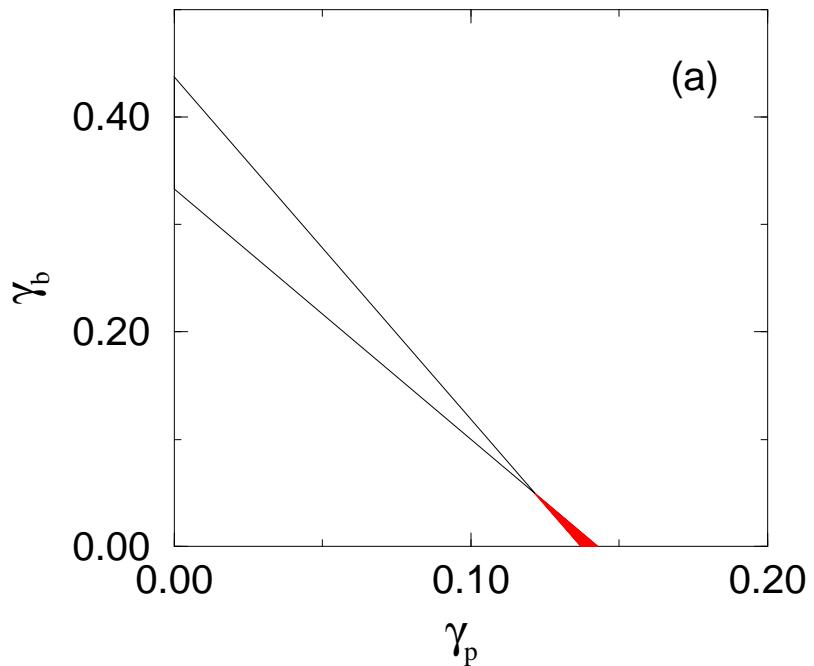


Figure 4
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